Wallis product, gamma function and n-dimensional spheres

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The aim of this article is to derive formulas for the volume and surface area of n-dimensional spheres. In this way, the Wallis product is calculated and some properties of the Gaussian gamma function are presented.

The Wallis product

As we will come back to this later, let's start by calculating the integrals

$$\int_0^{\pi/2} \sin^n x \, dx.$$

To do this, we first assume $n \geq 2$. Partial integration and trigonometric Pythagoras results in

$$\int_0^{\pi/2} \sin^{n-1} x \sin x \, dx = \Big[\sin^{n-1} x (-\cos x) \Big]_0^{\pi/2} - \int_0^{\pi/2} (n-1) \sin^{n-2} x \cos x (-\cos x) \, dx$$

$$= (n-1) \int_0^{\pi/2} \sin^{n-2} x \cos^2 x \, dx$$

$$= (n-1) \int_0^{\pi/2} \sin^{n-2} x \, dx - (n-1) \int_0^{\pi/2} \sin^n x \, dx.$$

If we move the last term to the other side, we get the recursion formula

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx.$$

Since

$$\int_0^{\pi/2} \sin^0 x \, dx = \frac{\pi}{2}$$

and

$$\int_0^{\pi/2} \sin^1 x \, dx = \Big[-\cos x \Big]_0^{\pi/2} = 1$$

is obtained for even $n = 2k$

$$\int_0^{\pi/2} \sin^{2k} x \, dx = \frac{2k-1}{2k} \cdot \frac{2k-3}{2k-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{2} \cdot \prod_{j=1}^{k} \frac{2j-1}{2j}$$

and for odd $n = 2k+1$

$$\int_0^{\pi/2} \sin^{2k+1} x \, dx = \frac{2k}{2k+1} \cdot \frac{2k-2}{2k-1} \cdots \frac{4}{5} \cdot \frac{2}{3} = \prod_{j=1}^{k} \frac{2j}{2j+1}.$$

This is actually all we need for later. But since we are already so close, let's go a little further and derive the aforementioned product formula.

Since for $0 \leq x \leq \pi/2$ $0 \leq \sin x \leq 1$ follows

$$\sin^{2k+1} x \leq \sin^{2k} x \leq \sin^{2k-1} x$$

and therefore also

$$\int_0^{\pi/2} \sin^{2k+1} x \, dx \leq \int_0^{\pi/2} \sin^{2k} x \, dx \leq \int_0^{\pi/2} \sin^{2k-1} x \, dx.$$

Inserting the product formulas just derived gives

$$\prod_{j=1}^{k} \frac{2j}{2j+1} \leq \frac{\pi}{2} \cdot \prod_{j=1}^{k} \frac{2j-1}{2j} \leq \prod_{j=1}^{k-1} \frac{2j}{2j+1}$$

and division by the right-hand product

$$\frac{2k}{2k+1} \leq \frac{\pi}{2} \cdot \frac{2k-1}{2k} \cdot \prod_{j=1}^{k-1} \frac{4j^2-1}{4j^2} \leq 1.$$

But because

$$\lim_{k\to\infty} \frac{2k}{2k+1} = \lim_{k\to\infty} \frac{2k-1}{2k} = 1$$

follows immediately

$$\frac{2}{\pi} = \lim_{k\to\infty} \prod_{j=1}^{k-1} \frac{4j^2-1}{4j^2} = \prod_{j=1}^{\infty} \frac{4j^2-1}{4j^2}.$$

This is the famous *Wallis product formula*.

The gamma function

The next step is to derive some properties of the gamma function. This was introduced in the article on the factorial as a generalization of the factorial to the real (and complex) numbers [1]. It was defined there as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt.$$

It was also shown that for $x > 0$ the *functional equation*

$$\Gamma(x+1) = x\Gamma(x)$$

[1] We only consider real x here.

is fulfilled. With $\Gamma(1) = \Gamma(2) = 1$, it follows that $F(x) = (x\ \mathit{1})!$ The functional equation immediately shows that the gamma function initially has poles at $x = 0$ and then further for all negative integers, which turn out to be odd. This is because the functional equation

$$\Gamma(x+2) = \Gamma((x+1)+1) = (x+1)\Gamma(x+1) = (x+1)x\Gamma(x)$$

and in general

$$\Gamma(x+n+1) = (x+n)(x+n-1) \cdots x\Gamma(x).$$

This shows that the gamma function is already fully determined by its values for $0 \leqslant x1$. Division gives us

$$\Gamma(x) = \frac{\Gamma(x+n+1)}{x(x+1) \cdots (x+n)},$$

from which one obtains for $x := \quad n$ the assertion about the pole positions is obtained.

We continue our investigations by slightly reshaping the definition equation. Because of

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

schreiben wir

$$\Gamma(x) = \lim_{n\to\infty} \int_0^n t^{x-1} \left(1 - \frac{t}{n}\right)^n dt.$$

Now partial integration leads to

$$\int_0^n t^{x-1} \left(1 - \frac{t}{n}\right)^n dt = \frac{t^x}{x} \left(1 - \frac{t}{n}\right)^n \Big|_0^n - \int_0^n \frac{t^x}{x} \left(1 - \frac{t}{n}\right)^{n-1} (-1)\, dt$$

$$= \frac{1}{x} \int_0^n t\, \mathit{1}^x - \frac{t}{n}^{n-1} dt.$$

This integral is partially integrated again

$$\int_0^n t^x \left(1 - \frac{t}{n}\right)^{n-1} dt = t\,\frac{x+1}{x+1} \left(1 - \frac{t}{n}\right)^n \Big|_0^n - \int_0^n \frac{t^{x+1}}{x+1} \left(1 - \frac{t}{n}\right)^{n-2} - \frac{n-1}{n}\, dt$$

$$= \frac{n-1}{n} \cdot \frac{1}{x+1} \cdot \int_0^n t^{x+1} \left(1 - \frac{t}{n}\right)^{n-2} dt,$$

so that for the original integral

$$\int_0^n t^{x-1} \left(1 - \frac{t}{n}\right)^n dt = \frac{n - 1}{n} \cdot \frac{1}{x \cdot (x+1)} \int_0^n t^{x+1} \left(1 - \frac{t}{n}\right)^{n-2} dt$$

is obtained. After a further n2 partial integrations, the interfering term has disappeared and you have

$$\int_0^n t^{x-1}\left(1-\frac{t}{n}\right)^n dt = \frac{(n-1)\cdot(n-2)\cdots 1}{n^{n-1}}\cdot\frac{1}{x\cdot(x+1)\cdots(x+n-1)}\int_0^n t^{x+n-1}\,dt$$

$$= \frac{(n-1)!}{n^{n-1}}\cdot\frac{1}{x\cdot(x+1)\cdots(x+n-1)}\,\frac{n^{x+n}}{x+n}$$

$$= \frac{n!\,n^x}{x\cdot(x+1)\cdots(x+n)}.$$

Thus the important *Gaussian product representation* follows

$$\Gamma(x) = \lim_{n\to\infty}\frac{n!\,n^x}{x\cdot(x+1)\cdots(x+n)}.$$

We now show that they can be derived directly from *Weierstrass' product representation*

$$\Gamma(x) = \frac{e^{-\gamma x}}{x}\cdot\prod_{k=1}^{\infty} e^{x/k}\left(1+\frac{x}{k}\right)^{-1}$$

with the *Euler-Mascheroni constant*

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)$$

results. For this we calculate

$$\Gamma(x) = \frac{e^{-\gamma x}}{x}\cdot\prod_{k=1}^{\infty} e^{x/k}\left(1+\frac{x}{k}\right)^{-1} = \frac{e^{-\gamma x}}{x}\cdot\lim_{n\to\infty}\left(\prod_{k=1}^{n} e^{x/k}\cdot\prod_{k=1}^{n} k\cdot\prod_{k=1}^{n}\frac{1}{k+x}\right)$$

$$= e^{-\gamma x}\cdot\lim_{n\to\infty}\left(\prod_{k=1}^{n} e^{x/k}\cdot n!\prod_{k=0}^{n}\frac{1}{k+x}\right)$$

$$= e^{-\gamma x}\cdot\lim_{n\to\infty}\left(\exp\left(x\sum_{k=1}^{n}\frac{1}{k}\right)\cdot n!\cdot\prod_{k=0}^{n}\frac{1}{k+x}\right)$$

$$= \lim_{n\to\infty}\left(\exp\left(x\left(\sum_{k=1}^{n}\frac{1}{k}-\gamma\right)\right)\cdot n!\cdot\prod_{k=0}^{n}\frac{1}{1+x}\right)$$

$$= \lim_{n\to\infty} n^x\cdot n!\prod_{k=0}^{n}\frac{1}{k+x},$$

which is already the claim.

For the last formula, we use the *partial fraction decomposition of the cotangent*

$$\pi\cot\pi x = \frac{1}{x}+\sum_{k=1}^{\infty}\frac{2x}{x^2-k^2}$$

back[2]. This results in the *Wallis product representation of the sine*[3]

$$\sin \pi x = \pi x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right).$$

On the one hand, differentiation results in

$$\frac{(\sin \pi x)'}{\sin \pi x} = \pi \cot \pi x$$

and on the other hand

$$\frac{\left(x \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2}\right)\right)'}{x \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2}\right)} = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{-2x}{k^2}\left(1 - \frac{x^2}{k^2}\right)^{-1} = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2x}{x^2 - k^2}.$$

This means that we have

$$\frac{(\sin \pi x)'}{\sin \pi x} = \frac{\left(x \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2}\right)\right)'}{x \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2}\right)}.$$

However, this is precisely the case when

$$\left(\frac{\sin \pi x}{x \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2}\right)}\right)' = 0$$

or

$$\sin \pi x = Cx \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right)$$

applies. Now, however

$$\lim_{x \to 0} \frac{\sin \pi x}{x} = \lim_{x \to 0} \frac{\pi \cos \pi x}{1} = \pi$$

and

$$\lim_{x \to 0} \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right) = 1,$$

from which $C = \pi$ and thus the assertion follows.

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[2] For a derivation, see the article on Fourier series.

[3] The result from the last section follows with $x = 1/2$.

Die Verbindung zur Gammafunktion lässt sich nun leicht herstellen:

$$\frac{1}{\Gamma(x)\Gamma(1-x)} = \lim_{n\to\infty} \left(\frac{1}{n^x \cdot n!} \prod_{k=0}^{n} (x+k) \cdot \frac{1}{n^{1-x} \cdot n!} \prod_{k=0}^{n} (1-x+k) \right)$$

$$= x \cdot \lim_{n\to\infty} \prod_{k=1}^{n} \frac{k^2 - x^2}{k^2} \cdot \lim_{n\to\infty} \frac{n+1-x}{n}$$

$$= x \cdot \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right) = \frac{\sin \pi x}{\pi}.$$

With $x = 1/2$, $\Gamma(1/2) = \sqrt{\pi}$ follows.

It will prove to be advantageous later if we know the values $\Gamma(n/2 + 1/2)$ and $\Gamma(n/2+1)$ for natural n. For this we need nothing more than the functional equation $\Gamma(x + 1) = x\Gamma(x)$. First, let n be even. Then

$$\Gamma\left(\frac{n+1}{2}\right) = \frac{\sqrt{\pi}}{2^{n/2}} \cdot \prod_{j=1}^{n/2} (2j-1) = \sqrt{\pi} \cdot \prod_{j=1}^{n/2} \frac{2j-1}{2}$$

und

$$\Gamma\left(\frac{n}{2}+1\right) = (n/2)! = \prod_{j=1}^{n/2} \frac{2j}{2}.$$

We prove this by induction. The formulas are correct for $n = 2$ because

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

and $\Gamma(2) = 1$. Let us now assume that they apply for $n - 2$. Then follows

$$\Gamma\left(\frac{n+1}{2}\right) = \Gamma\left(\frac{n-1}{2}+1\right) = \frac{n-1}{2} \cdot \Gamma\left(\frac{n-1}{2}\right)$$
$$= \frac{n-1}{2} \cdot \frac{\sqrt{\pi}}{2^{n/2-1}} \cdot \prod_{j=1}^{n/2-1} (2j-1) = \frac{\sqrt{\pi}}{2^{n/2}} \prod_{j=1}^{n/2} (2j-1)$$

and

$$\Gamma\left(\frac{n}{2}+1\right) = \frac{n}{2} \cdot \Gamma\left(\frac{n}{2}\right) = \frac{n}{2} \cdot \left(\frac{n-2}{2}\right)! = (n/2)!$$

according to the induction assumption. Now let n be odd. Then we have the formulas

$$\Gamma\left(\frac{n+1}{2}\right) = \left(\frac{n-1}{2}\right)! = \prod_{j=1}^{(n-1)/2} \frac{2j}{2}$$

and

$$\Gamma\left(\frac{n}{2}+1\right) = \frac{\sqrt{\pi}}{2^{(n+1)/2}} \cdot \prod_{j=1}^{(n+1)/2} (2j-1) = \sqrt{\pi} \cdot \prod_{j=1}^{(n+1)/2} \frac{2j-1}{2}$$

must be shown. However, these result directly from the substitutions $n \mapsto k-1$ or $n \mapsto k+1$ from those previously proven. All that remains is to check the first formula for $n = 1$, but this is trivial because $0! = 1$ is trivial[4].

Volume and surface area of the sphere in the $\mathbb{R}^n$

In general, the (Euclidean) n-dimensional sphere with radius r is defined as

$$K_n(r) = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \sum_{k=1}^{n} x_k^2 \le r^2 \right\}.$$

In the known cases $n = 1$, the interval $[-r, r]$ with length $V_1(r) = 2r$ is obtained, for $n = 2$ the circle with area $V_2(r) = \pi r^2$ and for $n = 3$ finally the sphere with volume $V_3(r) = 4/3\pi r^3$. Initially, there is no connection between n and the n-dimensional volume $V_n(r)$. Nevertheless, there is a closed formula. This is

$$V_n(r) = \frac{2\pi^{n/2}}{n\Gamma(n/2)} \cdot r^n = \frac{\pi^{n/2}}{\Gamma(n/2+1)} \cdot r^n.$$

We immediately see that it provides the correct result for the cases known to us. We now proceed to prove it by induction. First of all, we notice that from symmetry $V_n(r) = r^n V_n(1)$ and it is therefore sufficient to restrict ourselves to the volume of the unit sphere. So let the formula be correct for $n-1$. The unit sphere in $\mathbb{R}^n$ lies between $x_n = -1$ and $x_n = 1$. The planes $x_n =$ const. intersect in the range $-1 < x_n < 1$ with $K_n(1)$ in an $(n-1)$-dimensional sphere with radius $\sqrt{1-x_n^2}$. The volume of this sphere is by induction assumption

$$V_{n-1}\left(\sqrt{1-x_n^2}\right) = \frac{2\pi^{\frac{n-1}{2}}}{(n-1)\Gamma\left(\frac{n-1}{2}\right)} \cdot (1-x_n^2)^{(n-1)/2}.$$

The volume of the unit sphere is now simply

$$V_n(1) = \int_{-1}^{1} V_{n-1}\left(\sqrt{1-x_n^2}\right) dx_n = \frac{2\pi^{\frac{n-1}{2}}}{(n-1)\Gamma\left(\frac{n-1}{2}\right)} \int_{-1}^{1} (1-x_n^2)^{(n-1)/2}\, dx_n.$$

The substitution $x_n = \cos t$

$$\int_{-1}^{1} (1-x_n^2)^{(n-1)/2}\, dx_n = \int_{\pi}^{0} \sin^{n-1} t\,(-\sin t)\, dt = \int_{0}^{\pi} \sin^n t\, dt = 2\int_{0}^{\pi/2} \sin^n t\, dt.$$

[4]The empty product is by definition 1.

However, we have already calculated this integral. It is for even n

$$\int_0^{\pi/2} \sin^n t\, dt = \frac{\pi}{2} \cdot \prod_{j=1}^{n/2} \frac{2j-1}{2j}$$

and for odd n

$$\int_0^{\pi/2} \sin^n t\, dt = \prod_{j=1}^{(n-1)/2} \frac{2j}{2j+1}.$$

With the results of the last section, we can use these formulas in the equation

$$\int_0^{\pi/2} \sin^n dx = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$$

for any n. This results in the following for the volume

$$V_n(1) = \frac{2\sqrt{\pi^{n-1}}}{(n-1)\Gamma\left(\frac{n-1}{2}\right)} \cdot \sqrt{\pi} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} = 2\pi^{n/2} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{(n-1)\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}+1\right)}.$$

Now we are almost there. We are only writing

$$\begin{aligned}(n-1)\cdot\Gamma\left(\frac{n-1}{2}\right)\cdot\Gamma\left(\frac{n}{2}+1\right) &= (n-1)\cdot\Gamma\left(\frac{n-1}{2}\right)\cdot\frac{n}{2}\cdot\Gamma\left(\frac{n}{2}\right) \\ &= n\cdot\Gamma\left(\frac{n}{2}\right)\cdot\Gamma\left(\frac{n-1}{2}+1\right)\end{aligned}$$

and thus obtain the desired result

$$V_n(1) = \frac{2\pi^{n/2}}{n\Gamma(n/2)}.$$

Analytical values up to $n = 10$ and the numerical values for the unit sphere are given in Table 1.1. If we now have fun and plot $V_n(1)$ over real n, we obtain the curve in Figure 1.1. A maximum can be recognized at $n \approx 5.26$.

n	1	2	3	4	5	6	7	8	9	10
$V_n(r)$	$2r$	πr^2	$\frac{4}{3}\pi r^3$	$\frac{\pi^2}{2}r^4$	$\frac{8\pi^2}{15}r^5$	$\frac{\pi^3}{6}r^6$	$\frac{16\pi^3}{105}r^7$	$\frac{\pi^4}{24}r^8$	$\frac{32\pi^4}{945}r^9$	$\frac{1}{120}r^{10}$
$V_n(1)$	2,00	3,14	4,19	4,93	5,26	5,17	4,72	4,06	3,30	2,55

Table 1.1: The volume $V_n(r)$ of the n-dimensional sphere.

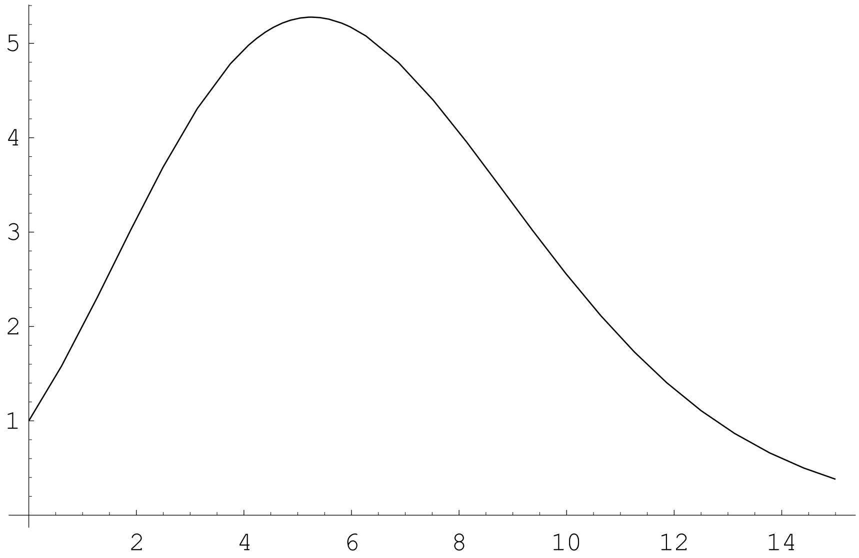

Figure 1.1: The volume $V_n(1)$ of the unit sphere over n.

However, it is surprising that the volume becomes smaller and smaller as n increases. In fact

$$\lim_{n\to\infty} V_n(1) = 0.$$

We show this with the help of the *Stirling formula*[5]

$$n! = \Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n .$$

This allows you to roughly estimate the volume as follows:

$$V_n(1) = \frac{2\pi^{n/2}}{n\Gamma(n/2)} \sim \frac{2\pi^{n/2}}{n\sqrt{\pi(n-1)}}\left(\frac{2e}{n-1}\right)^{(n-1)/2} \leq \frac{2\cdot 4^{n/2}}{n\sqrt{n}}\left(\frac{2\cdot 4}{n-1}\right)^{(n-1)/2}$$

$$= \frac{\sqrt{2^{2+2n+(n-1)+(2n-2)}}}{n\sqrt{n-1}^{\,n}} = \frac{1}{n}\cdot\frac{\sqrt{2^{5n-1}}}{\sqrt{n-1}^{\,n}}.$$

However, this is close to 0, as the first factor is close to 0 and the estimate for the second factor is

$$\sqrt{2^{5n-1}} \leq \sqrt{n-1}^{\,n}$$

$$\Leftrightarrow \quad (5n-1)\ln\sqrt{2} \leq n\ln\sqrt{n-1}$$

$$\Leftrightarrow \quad \ln\sqrt{2} \leq \frac{n}{5n-1}\ln\sqrt{n-1}$$

[5]The Stirling formula is derived in the article on the factorial.

is valid for sufficiently large n.

Let us now turn to the surface $O_n(r)$ of the *n-dimensional* sphere. The calculation of the surface area can be done using

$$V_n(r) = \int_0^r O_n(t)\, dt$$

to the problem already solved for the volume. The result is

$$O_n(r) = \frac{\partial V_n(r)}{\partial r} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \cdot r^{n-1}.$$

The first 10 values are summarized in Table 1.2. The continuous graph is shown in Fig. 1.2. The maximum has shifted to $n \approx 7.26$. However, the asymptotic behavior has not changed, i.e. the following also applies

$$\lim_{n\to\infty} O_n(1) = 0.$$

The proof is recommended as an exercise.

n	1	2	3	4	5	6	7	8	9	10
$O_n(r)$	2	$2\pi r$	$4\pi r^2$	$2\pi^2 r^3$	$\frac{8\pi^2}{3} r^4$	$\pi^3 r^5$	$\frac{16\pi^3}{15} r^6$	$\frac{\pi^4}{3} r^7$	$\frac{32\pi^4}{105} r^8$	$\frac{\pi^5}{12} r^9$
$O_n(1)$	2,00	6,28	12,57	19,74	26,32	31,01	33,07	32,47	29,69	25,50

Table 1.2: The surface area $O_n(r)$ of the n-dimensional sphere.

Calculation with spherical coordinates

You can calculate the surface of an *n-dimensional* sphere more quickly if you introduce sphere coordinates in $\mathbb{R}^n$. For these purposes, we don't even need to be interested in how this is actually done, it is enough to know that it is possible.

Let $|x|$ denote the Euclidean absolute value

$$|x|^2 = \sum_{k=1}^{n} x_k^2$$

of a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We now consider the integrals I_n in Cartesian

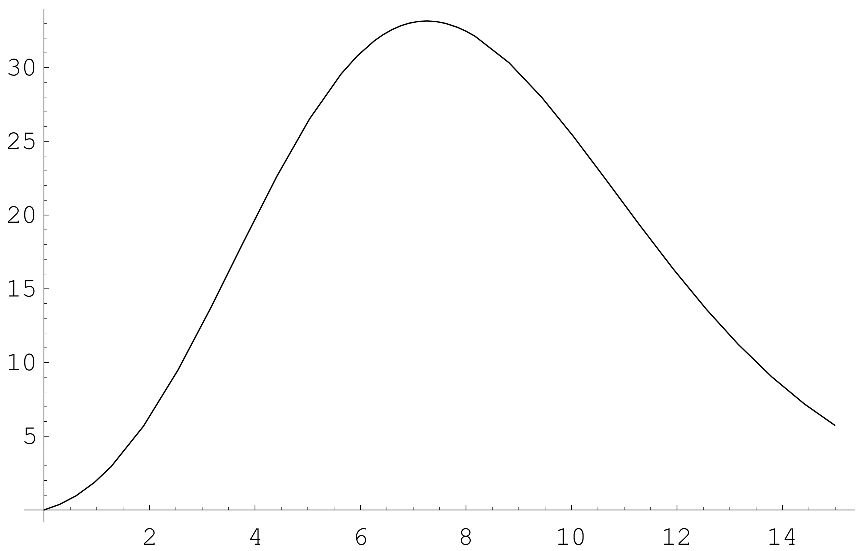

Figure 1.2: The surface $O_n(1)$ of the unit sphere over n.

Coordinates

$$I_n = \int_{\mathbb{R}^n} e^{-|x|^2} dx = \int_{x_1=-\infty}^{\infty} \cdots \int_{x_n=-\infty}^{\infty} \exp\left(-\sum_{k=1}^{n} x_k^2\right) dx_1 \cdots dx_n$$

$$= \int_{x_1=-\infty}^{\infty} \cdots \int_{x_n=-\infty}^{\infty} e^{-x_1^2} \cdots e^{-x_n^2} dx_1 \cdots dx_n$$

$$= \left(\int_{-\infty}^{\infty} e^{-x_1^2} dx_1\right) \cdots \left(\int_{-\infty}^{\infty} e^{-x_n^2} dx_n\right) = \left(\int_{-\infty}^{\infty} e^{-x_1^2} dx_1\right)^n = I_1^n$$

and in spherical coordinates

$$I_n = \int_{r=0}^{\infty} \int_{S_n} e^{-r^2} r^{n-1} \, dr \, d\Omega_n,$$

where S_n denotes the n-sphere and Ω_n the solid angle in $\mathbb{R}^n$. Now

$$O_n(1) = \int_{S_n} d\Omega_n$$

and therefore

$$I_n = O_n(1) \int_0^{\infty} e^{-r^2} r^{n-1} \, dr.$$

With the substitution $t = r^2$

$$\int_0^\infty e^{-r^2} r^{n-1}\, dr = \int_0^\infty e^{-t}\, t^{(n-1)/2} \frac{1}{2\sqrt{t}}\, dt = \frac{1}{2}\int_0^\infty e^{-t}\, t^{n/2-1}\, dt = \frac{\Gamma(n/2)}{2}$$

and therefore

$$I_n = O_n(1)\frac{\Gamma(n/2)}{2}.$$

Because $\Gamma(1) = 1$ and $O_2(1) = 2\pi$, $I_1 = \sqrt{I_2} = \sqrt{\pi}$. This also gives us the Result

$$O_n(1) = \frac{2}{\Gamma(n/2)} I_n = \frac{2}{\Gamma(n/2)} I_1^n = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$